Int J Theor Phys (2009) 48: 1872–1876
DOI 10.1007/s10773-009-9957-5

1-Soliton Solution of the Nonlinear Schrödinger's Equation with Kerr Law Nonlinearity Using Lie Symmetry Analysis

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Received: 26 November 2008 / Accepted: 27 January 2009 / Published online: 6 February 2009

Abstract This paper obtains the 1-soliton solution of the nonlinear Schrödinger's equation in a Kerr law media. The technique that is used to carry out the integration of this equation is the Lie symmetry analysis.

Keywords Solitons · Lie symmetry · Kerr law

1 Introduction

The nonlinear Schrödinger's equation (NLSE) is a very important equation in the area of Applied Mathematics, Theoretical Physics, Engineering Sciences and Biological Sciences [2, 6]. In particular, NLSE appears in the study of Fiber Optics and Bose-Einstein condensates. There are various kinds of solutions that are known for this equation. These include the periodic waves, doubly periodic waves, cnoidal waves, solitary waves and many more. There are various techniques that are used to integrate NLSE and obtain these kinds of solutions. The common methods that are frequently seen in various text books and research papers are the classical method of Inverse Scattering Transform (IST), Hirota's bilinear method and others. In this paper, however, a fairly less common method of integrability will be discussed to carry out the integration of the NLSE with Kerr law nonlinearity. The method is the usage of Lie symmetries. The basic idea of Lie symmetries is to study the invariance property of a given differential equation under continuous group of transformations. Lately, the symmetry analysis technique has been widely used to carry out the integration of many equations including the Lane-Emden equation [5] and quintic nonlinear

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Schrödinger equation [1] that are studied in the context of Astrophysics and nonlinear fiber optics, respectively.

The dimensionless form of the NLSE with Kerr law nonlinearity that is going to be studied in this paper is given by

$$iq_t + \frac{1}{2}q_{xx} + g(x)|q|^2 q = V(x)q. \tag{1}$$

In (1), the first term represents the evolution term, the second term is the dispersion term and $g(x)$ is the coefficient of the Kerr nonlinear term while $V(x)$ represents the potential. This equation is heavily studied in the context of the propagation of solitons through optical fibers for trans-oceanic and trans-continental distances [2, 6]. In this paper, the concentration is going to be in obtaining the localized explicit stationary solution to (1) of the form

$$q(x,t) = \phi(x)e^{-i\lambda t}, \tag{2}$$

where λ is a constant and the function ϕ depends on the variable x alone. Thus, from (1) and (2), $\phi(x)$ satisfies the time independent inhomogeneous nonlinear equation that is given by

$$\phi'' - 2g(x)(\phi)^3 + 2[\lambda - V(x)]\phi = 0. \tag{3}$$

The outline of the paper is as follows. In Sect. 2 we use the algorithm for computing Lie point symmetries of second-order ordinary differential equations and obtain symmetries of (3). Exact solution of NLSE is obtained in Sect. 3 and conclusions are given in Sect. 4.

2 Symmetry Analysis of NLSE

We now present a short overview of the use of Lie point symmetries in obtaining solutions of ordinary differential equations. The general procedure of finding Lie point symmetries of differential equations and their application to obtain analytic solutions of the equations can be found in many books (see for example, [3, 8, 9]). Lie group analysis is indeed the most powerful tool to find the general solution of ordinary differential equations.

If X, given by

$$X = \xi(x,\phi)\frac{\partial}{\partial x} + \eta(x,\phi)\frac{\partial}{\partial \phi} \tag{4}$$

is an admitted generator of (3), then

$$X^{[2]}\left(\frac{d^2\phi}{dx^2} - 2g(x)(\phi)^3 + 2[\lambda - V(x)]\phi\right)\Bigg|_{(3)} = 0, \tag{5}$$

where $X^{[2]}$ is the second prolongation of X and is given by

$$X^{[2]} = \xi(x,\phi)\frac{\partial}{\partial x} + \eta(x,\phi)\frac{\partial}{\partial \phi} + \eta^{(1)}(x,\phi,\phi')\frac{\partial}{\partial \phi'} + \eta^{(2)}(x,\phi,\phi',\phi'')\frac{\partial}{\partial \phi''} \tag{6}$$

with $\eta^{(k)}$, for $k = 1, 2$ being given by (see [3] or [8])

$$\eta^{(k)}(x, \phi, \phi', \phi'', \dots, \phi^k) = \frac{D\eta^{(k-1)}}{Dx} - \phi^k \frac{D\xi(x, \phi)}{Dx} \tag{7}$$

while $\eta^{(0)} = \eta(x, \phi)$ and the total derivative operator is defined as

$$\frac{D}{Dx} = \frac{\partial}{\partial x} + \phi' \frac{\partial}{\partial \phi} + \phi'' \frac{\partial}{\partial \phi'} + \dots + \phi^{(n+1)} \frac{\partial}{\partial \phi^{(n)}} + \cdots. \tag{8}$$

Condition (5) is known as the infinitesimal criterion of invariance. The application of the invariance condition to the differential equation (3) and the fact that the derivatives of the dependent variable are independent leads to an overdetermined system of four linear partial differential equations in ξ and η called the determining equations for the symmetry group of (3). The general solution of the determining equations gives the most general symmetry of the differential equation (3) [3, 4, 8, 9].

Expanding (5) yields

$$\begin{aligned} &-2\xi g'\phi^3 - 2\xi V'\phi - 6g\eta\phi^2 + 2\lambda\eta - 2V\eta + \eta_{xx} + (2\eta_{x\phi} - \xi_{xx})\phi' \\ &\quad + (\eta_{\phi\phi} - 2\xi_{x\phi})\phi'^2 - \xi_{\phi\phi}\phi'^3 + 2g\eta_\phi\phi^3 - 2\lambda\phi\eta_\phi + 2V\phi\eta_\phi - 4g\phi^3\xi_x \\ &\quad + 4\lambda\phi\xi_x - 4v\phi\xi_x - 4g\phi^3\phi'\xi_\phi + 4\lambda\phi\phi'\xi_\phi - 4V\phi\phi'\xi_\phi = 0. \end{aligned} \tag{9}$$

Since ξ and η are independent of ϕ', we may separate by powers of ϕ' and obtain the following linear overdetermined system of four partial differential equations:

$$\begin{aligned} &\phi'^3 : \xi_{\phi\phi} = 0, \\ &\phi'^2 : \eta_{\phi\phi} - 2\xi_{x\phi} = 0, \\ &\phi'^1 : 2\eta_{x\phi} - \xi_{xx} - 4g\xi_\phi\phi^3 + 4\lambda\phi\xi_\phi - 4V\phi\xi_\phi = 0, \\ &\phi'^0 : -2\xi g'\phi^3 - 2\xi V'\phi - 6g\eta\phi^2 + 2\lambda\eta - 2V\eta + \eta_{xx} + 2g\eta_\phi\phi^3 \\ &\qquad - 2\lambda\eta_\phi\phi + 2V\eta_\phi\phi - 4g\xi\phi^3 + 4\lambda\xi_x\phi - 4V\xi_x\phi = 0. \end{aligned}$$

Solving the above equations we conclude that the Lie point symmetry for (3) is of the form

$$X = a(x)\frac{\partial}{\partial x} + \left[\frac{1}{2}a'(x) + K\right]\phi\frac{\partial}{\partial \phi}, \tag{10}$$

subjected to the requirements

$$g(x) = \frac{g_0}{(a(x))^3}\exp\left[-2K\int_0^x \frac{1}{a(s)}ds\right] \tag{11}$$

and

$$a'''(x) - 4a(x)V'(x) - 8a'(x)[V(x) - \lambda] = 0, \tag{12}$$

where g_0 and K are arbitrary constants of integration.

3 Exact Solution of NLSE

We now use the above Lie point symmetry of (3) to construct canonical variables which will transform the inhomogeneous equation (3) to a cubic nonlinear Schrödinger equation without an external potential and with homogeneous nonlinearity.

Using the well known fact [7] that the invariance of the energy is associated with the translational invariance whose generator is of the form $X = \partial/\partial x$, one can define the canonical transformation related to the symmetry (10) as

$$r = h(x), \qquad u = f(x)\phi, \tag{13}$$

where $h(x)$ and $f(x)$ can be determined by requiring that $X = \partial/\partial r$ exists in the canonical variables to preserve the energy conservation law. This leads to

$$a(x)h'(x) = 1, \qquad a(x)f'(x) + \left[\frac{1}{2}a'(x) + K\right]f(x) = 0 \tag{14}$$

and on integration gives

$$h(x) = \int_0^x \frac{1}{a(s)}ds, \qquad f(x) = \frac{1}{\sqrt{a(x)}}\exp\left[-K\int_0^x \frac{1}{a(s)}ds\right]. \tag{15}$$

If we take $K = 0$, then our canonical transformation (13) becomes $r = \int_0^x (1/a(s))ds$, $u = a^{-1/2}(x)\phi$ and (3) is transformed to

$$\frac{d^2u}{dr^2} + Eu - 2g_0u^3 = 0, \tag{16}$$

where

$$E = -\frac{1}{4}(a'(x))^2 + \frac{1}{2}a(x)a''(x) + 2[\lambda - V(x)](a(x))^2. \tag{17}$$

If we multiply (16) by u', it can be immediately integrated to give a first-order ordinary differential equation, which can be reduced to the quadrature

$$r - r_0 = \pm\int_{u_0}^{u} \frac{ds}{(k_0 - Es^2 + g_0s^4)^{1/2}}, \tag{18}$$

where r_0 and k_0 are arbitrary constants of integration. Choosing $E = g_0 = -\mu^2$ in (18) yields the result

$$u = \frac{1}{\cosh \mu x} \tag{19}$$

which leads to the stationary 1-soliton solution of the NLSE with Kerr law nonlinearity.

4 Conclusions

This paper obtained the 1-soliton solution of the NLSE with Kerr law nonlinearity. The approach that was used to obtain the soliton solution is the Lie symmetry analysis approach. The Lie point symmetry analysis of the corresponding eigenvalue problem was carried out.

Subsequently using the canonical transformations and invariants, the corresponding soliton solution can be simply obtained by quadratures. In future, this problem will be extended to the cases of non-Kerr law nonlinearity.

Acknowledgement The research work of the second author (A.B.) was fully supported by NSF-CREST Grant No: HRD-0630388 and Army Research Office (ARO) along with the Air Force Office of Scientific Research (AFOSR) under the award number: W54428-RT-ISP and these supports are genuinely and sincerely appreciated.

The second author is also very grateful to Prof. C.M. Khalique and North-West University for the warm hospitality that he has received during his academic visit in November 2008.

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